Semi-Decentralized Approximation of an Optimal Control applied to a Cantilever Array

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Abstract—We illustrate a recently developed general theory of optimal control approximation through an application to cantilever arrays, as these encountered in Atomic Force Microscope Arrays. The theory applies to the field of finite length distributed systems where actuators and sensors are regularly spaced. It yields approximations implementable on semi-decentralized architectures.

I. INTRODUCTION

This work is a contribution to the area of semi-decentralized linear optimal control of large distributed systems with an application to the field of Arrays of Microsystems. A number of papers have been focused on the question "How to compute, in a decentralized manner, a distributed optimal control for a system comprised of a regular distribution of actuators and sensors". Most papers are dealing with infinite length systems, see [1] and [5] for systems governed by partial differential equations, and [2] for discrete systems.

In the previous papers [6] and [7], authors have shown how this question can be answered for a finite length beam endowed with a periodic distribution of piezoelectric sensors and actuators. The treatment was specific to this problem, and were suffering two main limitations. First, it was not clear how general control and observation operators could be taken into account by this approach. Second, the approximation was in the sense of high frequencies, so broad bandwidth controls were not covered by this strategy. In a recently submitted theoretical paper, we bring satisfactory solutions to both questions, and in the same time, we provide a well justified general mathematical framework based on a matrix functional calculus issued from spectral theory. In the present paper, we illustrate these theoretical results on one of the targeted application, namely control of cantilever arrays, encountered in Atomic Force Microscope Arrays.

We focus the presentation to the simple Linear Quadratic Regulator, because it is enough to fully explain the method. It can be extended to other optimal control theories for linear distributed systems. We underline that our effort, in this paper, is focused on illustrating the method more than providing a mathematically rigorous treatment. In the sequel, we begin with outlining a two-scale model of cantilever arrays. Then, all construction steps of the approximate Linear Quadratic Regulator are fully presented. Finally, we report numerical simulation results.

II. GENERAL NOTATIONS

The norm and the inner product of an Hilbert space $E$ are denoted by $\|\cdot\|_E$ and $(\cdot, \cdot)_E$. For a second Hilbert spaces $F$, $\mathcal{L}(E,F)$ denotes the space of continuous linear operators defined from $E$ to $F$. In addition, $\mathcal{L}(E,E)$ is denoted by $\mathcal{L}(E)$. One says that $\Phi \in \mathcal{L}(E,F)$ is an isomorphism from $E$ to $F$ if $\Phi$ is a one-to-one continuous mapping with a continuous inverse.

III. A TWO-SCALE MODEL OF CANTILEVER ARRAYS

We consider a one-dimensional cantilever array comprised of an elastic base, and a number of clamped elastic cantilevers with free end, see FIG 1. Assuming that the number of cantilevers is sufficiently large, an homogenized model was derived using a two-scale approximation method. This is reported in the detailed paper [3] devoted to static regime. The corresponding model extended to dynamic regime is introduced in the letter [4], and its validation is detailed in a paper which is in review for publication. The modelling papers were written in view of Atomic Force Microscopy application.

Fig. 1. Array of Cantilevers

The homogenized model was build within three steps. First, a change of variable is introduced so that to formulate the full model in a two-scale referential comprised of micro and macro variables. Then, it is approximated in the sense of large number of cantilevers. Finally, it is mapped back onto the natural referential and onto the actual system. The present control theory is developed on the model resulting from the second step, so it is expressed in the two-scale referential,
and a large but finite number of cantilevers is approximated by a distribution of an infinite number of cantilevers. After a number of simplifications, the approximate model expressed in the two-scale referential, appears as posed in a rectangle $\Omega = (0, L_B) \times (0, L_C)$, see FIG 2. The parameters $L_B$ and $L_C$ represent respectively the base length in the macroscale direction $x$ and the scaled cantilever length in the microscale variable $y$. The base is modelled by the line $\Gamma = \{(x, y) \mid x \in (0, L_B) \text{ and } y = 0\}$, and the rectangle $\Omega$ is filled by an infinite number of cantilevers. We describe the system motion by its bending displacement. The driving force of the actuator, the bending displacement, the bending coefficient and the width of the system motion by its bending displacement only. So, the reduction of the problem is simplified accordingly, so it can be replaced by any other realistic force distribution. Denoting by $u_B(t, x)$, $\rho^B$, $R^B$, and $\ell_C$, the base governing equation states

$$\rho^B \ddot{w}_B + R^B \dddot{w}_B = -\epsilon R^C \partial_{yyy}^3 w_C + u_B. \quad (1)$$

The base is assumed to be clamped, so the boundary conditions are

$$w_B = \dot{w}_B = 0 \quad (2)$$

at both ends. Cantilevers are oriented in the $y$-direction, and their motions are governed by an infinite number of Euler-Bernoulli equations distributed along the $x$-direction. Each cantilever is subjected to a control force $u_C(t, x)$, taken, for simplicity, constant along cantilevers. This choice does not affect the method presented hereafter, so it can be replaced by any other realistic force distribution. Denoting by $w_C(t, x, y)$, $\rho^C$ and $R^C$ cantilever bending displacements, mass per unit length, and bending coefficient, the governing equation in $(x, y) \in \Omega$ is

$$\rho^C \ddot{w}_C + R^C \dddot{w}_C = u_C, \quad (3)$$

endoewed with the boundary conditions

$$\begin{cases} w_C = w_B \text{ and } \dot{w}_C = 0 & \text{at } y = 0 \\ \dddot{w}_C = \partial_{yyy}^3 w_C = 0 & \text{at } y = L_C \end{cases} \quad (4)$$

representing an end clamped in the base, and a free end. Finally, both equations are supplemented with initial conditions on displacements and velocities,

$$w_B = w_{B,0}, \dot{w}_B = w_{B,1}, \quad w_C = w_{C,0}, \text{ and } \dot{w}_C = w_{C,1}.$$

The LQR problem is set for control variables $(u_1, u_2) \in U = L^2(\Gamma)^2$ and for the cost functional

$$\mathcal{J}(w_{B,0}, \dot{w}_{B,1}, w_{C,0}, \dot{w}_{C,1}; u_B, u_C) = \int_0^1 \left( \|\partial_x^2 w_B\|_{L^2(\Gamma)}^2 + \|\partial_x^2 w_C\|_{L^2(\Omega)}^2 + \|u_B\|_{L^2(\Gamma)}^2 + \|u_C\|_{L^2(\Gamma)}^2 \right) dt, \quad (5)$$

IV. MODEL REFORMULATION

The first step, in applying the method, consists in transforming the control problem into a control problem into another problem with internal distributed control and observation. To do so, we are lead to make additional assumptions yielding model simplifications. We set $\tilde{w}_C = w_C - w_B$, solution of an Euler-Bernoulli equation in cantilevers with homogeneous boundary conditions

$$\begin{cases} \rho^C \ddot{\tilde{w}}_C + R^C \dddot{\tilde{w}}_C = u_C - \rho^C \ddot{w}_B & \text{in } \Omega, \\ \tilde{w}_C = \dot{\tilde{w}}_C = 0 & \text{at } y = 0, \\ \partial_{xx}^2 \tilde{w}_C = \partial_{yyy}^3 \tilde{w}_C = 0 & \text{at } y = L_C. \end{cases} \quad (6)$$

We introduce the basis of normalized eigenfunction $(\psi_k)_k$, solution of the corresponding eigenvalue problem

$$\begin{cases} \ddot{\psi}_k + \lambda_k \psi_k = 0 & \text{in } (0, L_C), \\ \psi_k(0) = \psi_k(L_C) = 0, \\ \partial_{xx}^2 \psi_k(L_C) = \partial_{yy}^3 \psi_k(L_C) = 0, \\ \|\psi_k\|_{L^2(0, L_C)} = 1. \end{cases} \quad (7)$$

It is well known that, in most practical applications, a very small number of cantilever modes is sufficient to properly describe the system. For the sake of simplicity, we take into account only the first one, keeping in mind that the method can handle more one mode. Therefore, we adopt the approximation

$$\tilde{w}_C(t, x, y) \simeq \psi_1(t) \psi_1(y),$$

where $\psi_1^C$ is the coefficient of the first mode $\psi_1$ in the modal decomposition of $w_C$. Introducing the mean $\bar{\psi}_C = \int_0^1 \psi_1 dy$, and $u_{C} = \int_0^{L_C} u_C dy$, we find that $\tilde{w}_C$ is solution of

$$\rho^C \ddot{\tilde{w}}_C + R^C \dddot{\tilde{w}}_C = u_C - \rho^C \ddot{w}_B. \quad (8)$$

In order to avoid the term $\ddot{w}_B$, we introduce $\tilde{w}_C = w_C^1 + \psi_1 w_B$, so as to make $\tilde{w}_C$ be solution of

$$\rho^C \ddot{w}_C^1 + R^C \dddot{w}_C^1 = u_C - \rho^C \ddot{w}_B. \quad (9)$$

Since,

$$\ddot{w}_C = \ddot{w}_C^1 + \psi_1 \ddot{w}_B = \ddot{w}_C^1 + \psi_1 \psi_1 \psi_1 = \ddot{w}_C^1 \psi_1 \tilde{w}_C,$$

we set $c_1 = \ddot{w}_C^1 \psi_1 (0)$, and obtain that the couple $(w_B, \tilde{w}_C)$ is solution of the system of equations posed on $\Gamma$,

$$\begin{cases} \rho^C \ddot{w}_B^1 + R^C \dddot{w}_B^1 = \lambda_1 \tilde{w}_C \quad \text{in } \Gamma, \\ \rho^C \ddot{w}_C^1 + R^C \dddot{w}_C^1 = \lambda_1 \tilde{w}_C \quad \text{in } \Gamma, \end{cases} \quad (10)$$

with the boundary conditions (2). The cost functional is simplified accordingly,

$$\mathcal{J} \simeq \int_0^1 \left( \|\partial_x^2 w_B(t, x)\|_{L^2(\Gamma)}^2 + \|\lambda_1 \tilde{w}_C(t, x)\|_{L^2(\Gamma)}^2 + \|u_B\|_{L^2(\Gamma)}^2 + \|u_C\|_{L^2(\Gamma)}^2 \right) dt,$$
V. Classical Formulation of the LQR Problem

Now, we write the above LQR problem in a classical abstract setting, see [8]. We set $z^T = (w_b \ w_c \ \bar{w}_b \ \bar{w}_c)$ the state variable, $u^T = (u_b \ u_c)$ the control variable, $A = \begin{pmatrix} -\rho^2 \delta^2_z & -\rho^2 \delta^2_c & 0 & 0 \\ -\delta^2_z \rho & -\delta^2_c \rho & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ the state operator, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \rho \end{pmatrix}$ the control operator, $C = \begin{pmatrix} \delta^2_z & 0 & 0 & 0 \\ 0 & \delta^2_c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ the observation operator, $S = I$ the weight operator and the functional $J(z_0, u) = \int_0^T \|Cz\|^2 + \langle \dot{S}u, u \rangle \, dt$. Consequently, the LQR problem, consisting in minimizing the functional under the constraint (8), may be written under its usual form as

$$\frac{dz}{dt}(t) = Az(t) + Bu(t) \quad \text{for } t > 0 \text{ and } z(0) = z_0,$$

minimize $f(z_0, u)$.

Here, $A$ is the infinitesimal generator of a continuous semigroup on the separable Hilbert space $Z = H^2_\rho (\Gamma) \times L^2 (\Gamma)^3$ with dense domain $D(A) = H^2_\rho (\Gamma) \cap H^2_\rho (\Gamma) \times H^2_\rho (\Gamma) \times L^2 (\Gamma)$. It is known that the control operator $B \in \mathcal{L}(U, Z)$, the observation operator $C \in \mathcal{L}(Z, Y)$, and $S \in \mathcal{L}(U, U)$, where $Y = L^2 (\Gamma)^3$. We also know that $(A, B)$ is stabilizable and that $(A, C)$ is detectable, in the sense that there exist $G \in \mathcal{L}(Z, U)$ and $F \in \mathcal{L}(Y, Z)$ such that $A - BG$ and that $A - FC$ are the infinitesimal generators of two uniformly exponentially stable continuous semigroups. It follows that for each $z_0 \in Z$, the LQR problem (10) admits a unique solution

$$u^* = -Kz$$

where $K = S^{-1}B^TPZ$, and $P \in \mathcal{L}(Z)$ is the unique self-adjoint nonnegative solution of the operational Riccati equation

$$(A^*P + PA - PBS^{-1}B^*P + C^*C) z = 0,$$

for all $z \in D(A)$. The adjoint $A^*$ of the unbounded operator $A$ is defined from $D(A^*) \subset Z$ to $Z$ by the equality $(A^* z, z)'_Z = (z, A^* z)'_Z$ for all $z \in D(A^*)$, and $z \in D(A)$. The adjoint $B^* \in \mathcal{L}(Z, U)$ of the bounded operator $B$ is defined by $(B^* z, u)_Y = (z, Bu)_Z$, the adjoint $C^* \in \mathcal{L}(Y, Z)$ being defined similarly.

VI. Semi-Decentralized Approximation

This section is devoted to formulate, step by step, the method of approximation.

A. Matrices of Functions of a Self-Adjoint Operator

Since the approximation method of $P$ is based on the concept of matrices of functions of a self-adjoint operator, this section is devoted to their definition. We discuss only the simplest case of compact operators which avoid spectral theory technicalities, because it is enough for the present example, see [9] for the general theory. From now on, we denote by $X$ the separable Hilbert space $L^2 (\Gamma)$ and by $\Lambda$ the self-adjoint operator $(\delta^4_{x\cdot \cdot \cdot})^{-1}$ with domain $D(\Lambda) = H^2 (\Gamma) \cap H^2_\rho (\Gamma)$ in $X$.

As $A$ is self-adjoint and compact, its spectrum $\sigma(\Lambda)$ is discrete, bounded and made up of real eigenvalues $\lambda_k$. They are solutions of the eigenvalue problem $\Lambda \Phi_k = \lambda_k \Phi_k$ with $\|\Phi_k\|_X = 1$. In the sequel, $I_\sigma = (\sigma_{\min}, \sigma_{\max})$ refers to an open interval that includes $\sigma(\Lambda)$.

For a given real valued function $f$, continuous on $I_\sigma$, $f(\Lambda)$ is the linear self-adjoint operator on $X$ defined by

$$f(\Lambda) z = \sum_{k=1}^{\infty} f(\lambda_k) \Phi_k,$$

where $z_k = (z_k, \Phi_k)_X$, with domain

$$D(f(\Lambda)) = \{ z \in X | \sum_{k=1}^{\infty} |f(\lambda_k)z_k|^2 < \infty \}.$$

Then, if $f$ is a $n_1 \times n_2$ matrix of real valued functions $f_{ij}$, continuous on $I_\sigma$, $f(\Lambda)$ is a matrix of linear operators $f_{ij}(\Lambda)$ with domain

$$D(f(\Lambda)) = \{ z \in X^{n_2} | \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} |f_{ij}(\lambda_k)z_k|^2 < \infty \ \forall i = 1..n_1 \}.$$

B. Factorization of $K$ by a Matrix of Functions of $\Lambda$

The second step in the semi-decentralized control approximation method is the factorization of $K$ under the form of a product of a function of $\Lambda$ with operators admitting a natural semi-decentralized approximation. To do so, we introduce three isomorphisms $\Phi_z \in \mathcal{L}(X^4, Z)$, $\Phi_U \in \mathcal{L}(X^2, U)$, and $\Phi_T \in \mathcal{L}(X^4, Y)$ mapping a powering of $X$ into $Z$, $U$, and $Y$ respectively, so that

$$a(\Lambda) = \Phi_Z^{-1} A \Phi_Z, \ b(\Lambda) = \Phi_Z^{-1} B \Phi_U, \ c(\Lambda) = \Phi_Y^{-1} C \Phi_Z, \ s(\Lambda) = \Phi_U^{-1} S \Phi_U$$

be some matrices of functions of $\Lambda$. In the present example, we propose

$$\Phi_Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \Phi_U = I, \ \text{and} \ \Phi_Y = \begin{pmatrix} \delta^2_x \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
where \( q(\lambda) = s^{-1}(\lambda) b^T(\lambda) p(\lambda) \), and where for all \( \lambda \in \sigma \), \( p(\lambda) \) is the unique self-adjoint nonnegative matrix solving the algebraic Riccati equation
\[
\begin{align*}
  a^T(\lambda) p + pa(\lambda) - pb(\lambda) s^{-1}(\lambda) b^T(\lambda) p \\
  + c^T(\lambda) c(\lambda) = 0.
\end{align*}
\] (13)

**Sketch of the proof** The algebraic Riccati equation can be found after replacing \( A, B, C \) and \( S \) by their decomposition in the Riccati equation (12).

In the sequel, we require that the algebraic Riccati equation (13) admits a unique solution for all \( \lambda \in I_\sigma \) which is checked numerically.

**Remark 2** : In this example, \( \Phi_U \) and \( \Phi_Z \) are some matrices of functions of \( \Lambda \), and so is \( K \),
\[
  K = k(\Lambda).
\] (14)

Thus, the approximation is developed directly on \( k(\Lambda) \), but we emphasize that in more generic situations it is pursued on \( q(\Lambda) \).

**Remark 3** : Introducing the isomorphisms \( \Phi_Z, \Phi_Y, \) and \( \Phi_T \) allows to consider a broad class of problems where the operators \( A, B, C \) and \( S \) are not strictly functions of a same operator. In this particular application, the observation operator \( C \) is composed with the operator \( \partial_2^2 \). This is taken into account in \( \Phi_T \) in a manner in which \( \Phi_T \) is a function of \( \Lambda \) only.

**Remark 4** : We indicate how the isomorphisms \( \Phi_Z, \Phi_Y, \) and \( \Phi_U \) have been chosen. The choice of \( \Phi_Z \) comes directly from the expression of the inner product \( (z, z')_Z = (\Phi_Z^{-1} z, \Phi_Z^{-1} z')_X \) and from \( (z_1, z'_1)_H_0(\Gamma) = (X^2)z_1, (X^2)z'_1 \) \( L^2(\Gamma) \). For \( \Phi_T \), we start from \( C = \Phi_T C(\Lambda) \Phi_T^{-1} \) and from the relation \( (y, y')_T = (\Phi_T y, \Phi_T y')_X \) which implies that \( \partial_2^2 = (\Phi_T)^H_1, \partial_1(\Lambda) \Lambda^{-\frac{1}{2}} \) and \( \Lambda^T(\Gamma) = (\Phi_T)^H_2 \). The expression of \( \Phi_T \) follows. Choosing \( \Phi_U \) is straightforward.

**C. Approximation of the Functions of \( \Lambda \)**

The third step in the method consists in an approximation of a general function of \( \Lambda \) by a simpler function of \( \Lambda \) easily discretized and implemented in a semi-decentralized architecture. The strategy must be general, and in the same time the approximation must be accurate. A simple choice would be to adopt a polynomial or a rational approximation, but their discretization yields very high errors due to the powers of \( \Lambda \). This can be avoided when using the Dunford-Schwarz formula, see [10], representing a function of an operator, because it involves only the operator \( (\Gamma I - \Lambda)^{-1} \) which may be simply, and accurately approximated. However, this formula requires the function to be holomorphic inside an open vicinity of \( \sigma \). Since the function is generally not known, this set cannot be easily determined, so we prefer to proceed within two steps. First, the function is approximated through a highly accurate rational approximation, then the Dunford-Schwarz formula is applied to the rational approximation, with a path tracing out an ellipse including \( I_\sigma \) but no poles.

Since the interval \( I_\sigma \) is bounded, each function \( k_{ij}(\lambda) \) have a rational approximation over \( I_\sigma \), that we write under a global formulation,
\[
  k_N(\lambda) = \frac{\sum_{m=0}^{N} d_m \lambda^m}{\sum_{m=0}^{N} d_m^* \lambda^m},
\] (15)

where \( d_m, d_m^* \) are matrices of coefficients and \( N = (N^N, N^D) \) is the couple comprised of the matrice \( N^N \) of numerator polynomial degrees and the matrice \( N^D \) of denominator polynomial degrees. The path \( \gamma \), in the Dunford-Schwarz formula,
\[
  k_N(\lambda) = \frac{1}{2\pi i} \int_{\gamma} k_N(\xi) (\xi I - \Lambda)^{-1} d\xi,
\]

is chosen to be an ellipse parametrized by
\[
  \xi(\theta) = \xi_1(\theta) + i\xi_2(\theta), \text{ with } \theta \in [0, 2\pi].
\]

The parametrization is used as a change of variable, so the integral is rewritten on the form \( I(g) = \int_{\gamma} g(\xi) d\theta \), and may be approximated by a quadrature formula involving \( M \) nodes \( (\theta_l)_{l=1,...,M} \in [0, 2\pi], \) and \( M \) weights \( (w_l)_{l=1,...,M} \).
\[
  I_M(g) = \sum_{l=1}^{M} g(\theta_l) w_l.
\]

For each \( z \in X^4 \) and \( \zeta \in \gamma \), we introduce the four-dimensional vector field
\[
  v^\zeta = -i\zeta' k_N(\zeta)(\xi I - \Lambda)^{-1} z.
\]

Decomposing \( v^\zeta \) into its real part \( v_r^\zeta \) and its imaginary part \( v_i^\zeta \), the couple \( v_r^\zeta, v_i^\zeta \)' is solution of the system
\[
\begin{align*}
  \zeta_1 v_r^\zeta - \zeta_2 v_i^\zeta + A v_r^\zeta &= Re(-i\zeta' k_N(\zeta)) z, \\
  \zeta_2 v_r^\zeta + \zeta_1 v_i^\zeta + A v_i^\zeta &= Im(-i\zeta' k_N(\zeta)) z.
\end{align*}
\] (16)

Thus, combining the rational approximation \( k_N \) and the quadrature formula yields an approximate realization \( k_{N,M}(\Lambda) \) of \( k(\Lambda) \),
\[
  k_{N,M}(\Lambda) z = \frac{1}{2\pi} \sum_{l=1}^{M} v_l(\theta_l) w_l.
\] (17)

This formula is central in the method, so it is the center of our attention in the simulations. A fundamental remark is that, a "real-time" realization, \( k_{N,M}(\Lambda) z \), requires solving \( M \) systems like (16) corresponding to the \( M \) nodes \( \zeta_l(\theta_l) \). The matrices \( k_N(\zeta_l(\theta_l)) \) could be computed "off-line" once and for all, and stored in memory, so their determination would not penalize a rapid real-time computation. In total, the ultimate parameter responsible of accuracy in a real-time computation, apart from spatial discretization discussed in next Section, is \( M \) the number of quadrature points.

**D. Spatial Discretization**

The final step consists in a spatial discretization of Equation (16), it does not represent a specific novelty, so we do not discuss it through numerical simulations. For the sake of simplicity, the interval \( \Gamma \) being meshed with regularly spaced
nodes separated by a distance \( h \), we introduce \( \Lambda_{h}^{-1} \) the finite difference discretization of \( \Lambda^{-1} \),

\[
\begin{pmatrix}
\frac{h^2}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-\frac{h^2}{2} & 2h & -\frac{h^2}{2} & 0 & 0 & 0 & \cdots & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & \cdots & 0 & 0 & 0 & -\frac{h^2}{2} & 2h & -\frac{h^2}{2} \\
0 & \cdots & 0 & 0 & 0 & 0 & h^2 \\
\end{pmatrix}
\]

\( \Lambda_{h}^{-1} = \frac{1}{h} \)

In practice, the discretization length \( h \) is chosen small compared to the distance between cantilevers. Then, \( z_{b} \) denoting the vector of nodal values of \( z \), for each \( \zeta \) we introduce \( (v_{1,b}^{z}, v_{2,b}^{z}) \), a discrete approximation of \( (v_{1}^{z}, v_{2}^{z}) \), solution of the discrete set of equations,

\[
\begin{align*}
\zeta v_{1,b}^{z} - \zeta v_{2,b}^{z} + \Lambda_{h} v_{1,b}^{z} &= Re (-i \zeta k_{N}(\zeta)) z_{b}, \\
\zeta v_{2,b}^{z} + \zeta v_{1,b}^{z} + \Lambda_{h} v_{2,b}^{z} &= Im (-i \zeta k_{N}(\zeta)) z_{b}.
\end{align*}
\]

Finally, in an approximate optimal control, intended to be implemented in a set of spatially distributed actuators, could be estimated from the nodal values,

\[
k_{N,M,b} = \frac{1}{2\pi} \sum_{l=1}^{M} \zeta_{l},
\]

estimated at mesh nodes.

VII. NUMERICAL VALIDATION

To build a rational interpolation \( k_{N} \) of the form (15) over \( I_{\sigma} \), we mesh the interval with \( L+1 \) distinct nodes \( \lambda_{0}, \ldots, \lambda_{L} \). Then all \( p(\lambda_{m}) \) solutions of the algebraic Riccati equation are accurately computed with a standard solver. Computing the rational approximation start by imposing \( L+1 \) conditions

\[
k_{N}(\lambda_{m}) = k(\lambda_{m}),
\]

or equivalently that

\[
\sum_{m=0}^{N} d_{m} \lambda_{m}^{n} - k(\lambda_{m}) \sum_{m'=0}^{N} d_{m}' \lambda_{m}' = 0,
\]

for \( n = 0, \ldots, L+1 \). Then, when \( L \) is large enough, the resulting system with \( N^{N} + N^{D} + 2 \) unknowns, \( [d, d'] = [d_{0}, \ldots, d_{N^{N}}, d_{0}', \ldots, d_{N^{D}}]' \), is overdetermined, so it is solved in the mean square sense.

In a numerical experiment, we have set all coefficients \( R^B \), \( \rho^B \), \( \ell_{C} \), \( R^C \), \( \rho^C \) and \( L^1 \) to one, and \( L^2 = 4.73 \). Thus, all eigenvalues of \( \Lambda \) turns to be included in \((0, 1)\), the first cantilever eigenvalue turns to be equal to \( \lambda_{1}^{C} = 12.36 \), \( \psi_{1} = -0.78 \) and \( c_{1} = 9.68 \). Moreover, we have chosen \( L = 100 \) nodes logarithmically distributed along \( I_{\sigma} = (10^{-2}, 1) \). We remark that the shapes of all spectral functions \( k_{ij} \) involved in \( K \), represented in Figure 3, exhibit a singular behaviour at the origin. This shows that this example is by no means trivial. In Table I, we report polynomial degrees \( N = (N^{N}, N^{D}) \) and relative errors

\[
e_{ij} = \frac{||k_{ij,N} - k_{ij}||_{L^2(I_{\sigma})}}{||k_{ij}||_{L^2(I_{\sigma})}},
\]

between the exact \( k \) and its rational approximation \( k_{N} \). The degrees \( N^{N} \) and \( N^{D} \) can be chosen sufficiently large so that errors are sufficiently small, since this has no effect on online control computation time.

Numerical integrations have been performed with a standard trapezoidal quadrature rule. Relative errors, between the exact functions and final approximations,

\[
E_{ij} = \frac{||k_{ij,N,M} - k_{ij}||_{L^2(I_{\sigma})}}{||k_{ij}||_{L^2(I_{\sigma})}},
\]

are reported in Figure 4, in logarithmic scale, for \( M \) varying from 10 to \( 10^{7} \). The results are satisfactory. Accuracy is proportional to the number of nodes. So it may be easily tuned without changing spatial complexity governed by the operator \( \Lambda \). Nevertheless, A lack of precision is observed for the function \( k_{23} \), which is due to its highly singular behaviour at zero. Refinements of the quadrature procedure are required to handle with such singularity.

**REFERENCES**

Fig. 4. Errors between $k$ and $k_{N,M}$


