Approximations of distributed control operators on quasi-local architectures

M. Lenczner, G. Montseny and Youssef Yakoubi

Abstract

This paper is focused on the derivation of state-realizations of diffusive type for linear operators solutions of some partial differential operational equations. The practical interest of this work relates for example to the context of intensive computation in embedded systems when embedded computers have a semi-decentralized architecture such as, for example, Cellular Neural Networks.

1 Introduction

Many partial differential problems can be straightforwardly solved by means of networks of interconnected computers such as Cellular Neural Networks (CNN), which consist of arrays of analogic processors with connections limited to immediate neighbors (see [2] for details). Today’s available CNN’s chips have about $128 \times 128$ analogic programmable cells, which enables to solve systems of (possibly non-linear) differential equations with as many unknowns. Additional basic instructions about local simple algebraic operations such as additions or multiplications are also available, so, the realization of finite differences operators is an easy task with such architectures. The huge advantage of the CNN computers is that basic instructions operate on a very large amount of data in an extremely short time (a few nanoseconds).

In this note, we address the problem of the concrete realization of linear operators on infinite dimensional spaces in a way that will be compatible with real time and embedded computation, i.e. that requires only the few possible operations already mentioned.

The concept of “diffusive representation” [5] reveals itself well-suited to the state-space realization of linear integral operators. Considering functions $u$ defined on a one-dimensional domain and a causal kernel operator $P$, the diffusive realization of $Pu$ takes the following integral form involving the resolvent of operator $\partial_x$ in the algebra of causal linear bounded operators on $L^2$:

$$ (Pu)(x) = \int \mu(x, \xi)(\partial_x + \gamma(\xi)I)^{-1}u(x)d\xi. $$

(1)
So, when \( \text{supp} \ u \subset [a, b] \), the operator \( u \mapsto Pu \) can be realized by solving the following Cauchy problem with null initial condition: \( \partial_x \psi + \gamma(\xi) \psi = u, \ \xi \in \mathbb{R} \) and \( x \in [a, b] \). Approximations at nodes \( x_i \) then require to solve a diagonal differential system: \( \partial_x \psi_n + \gamma(\xi_n) \psi_n = u_n \), in such a way that \( \tilde{P}u(i) \approx \sum_n \mu_n(x_i) \psi_n(x_i) \). In the general case, 1D-operators can similarly be realized by decomposing them in their causal and anti-causal parts, each of them having a representation of the form (1). Thanks to its local formulation, the resulting numerical realization is implementable on CNN architectures.

In this note, there is the first attempt to build diffusive realizations of operators that are themselves the unknowns of operational equations. Our main contribution consists of the determination of the equations verified by the so-called diffusive symbol \( \mu \). We also give a sufficient characterization of the admissible paths \( \gamma \). All the results announced for the one-dimensional case may be extended to higher dimensional problems. Other methods have already been proposed in [1], [3] and [4] for the resolution of the same kind of problems. The two first approaches are limited to space invariant equations posed on infinite domains when the third provides a high frequency approximation and is limited to a class of operational equations.

2 Diffusive realization of integral operators

We denote \( \omega := ]0,1[ \) and \( \Omega := \omega \times \omega \). Throughout this note, we shall use the uperscripts \( + \) or \(-\) to refer to causal or anti-causal operators, and the convention \( \mp = -(+\)\).

We consider an operators \( P \) in \( L^2(\omega) \) formulated under the general integral form

\[
(Pu)(x) = \int_{\omega} p(x, y) u(y) \, dy.
\]

An operator \( P \) is said to be causal (respectively anti-causal) if \( p(x, y) = 0 \) for \( y > x \) (respectively for \( y < x \)). Diffusive realizations of \( P \) are based on its unique decomposition into causal and anti-causal parts: \( P = P^+ + P^- \), where \( (P^+ u)(x) = \int_x^\infty p(x, y) u(y) \, dy \) and \( (P^- u)(x) = \int_x^{y} p(x, y) \, u(y) \, dy \). The so-called impulse response \( \tilde{p} \) is defined by \( p(x, y) = \tilde{p}(x, x - y) \). The variables \( x \) and \( y \) are treated on an unequal footing, assuming that the causal (resp. anti-causal) impulse response is analytic with respect to \( y \), with analytic extension to \( \mathbb{R}^+ \) (resp. \( \mathbb{R}^- \)) locally integrable, and that for each \( y \), the function \( x \mapsto p(x, y) \) belongs to \( L^2(\omega) \).

For given \( a^\pm \in \mathbb{R} \), we consider \( \xi \mapsto \gamma^\pm(\xi) \) two complex Lipschitz functions from \( \mathbb{R} \) to \( [a^+, +\infty[+i\mathbb{R} \subset \mathbb{C} \) such that \( |\gamma^\pm| \geq b > 0 \) almost everywhere which define simple arcs closed at infinity. Moreover we assume that they are included in some sector \( k + e^{i(-\alpha, +\alpha)} \mathbb{R}^+ \) with \( 0 \leq \alpha < \frac{\pi}{2} \).

From now on, we use the convenient notation, \( \langle \mu, \psi \rangle := \int_{\mathbb{R}} \mu(\xi) \psi(\xi) \, d\xi \) and we note that in the case where \( \mu \) would not be a locally integrable function, a more general duality product, to be specified in each concrete case, would be involved in place of the integral.

Consider also the so-called \( \gamma^\pm \)-representations of \( u \), denoted by \( \psi^\pm(u) \) and defined as the unique solutions of the following direct and backward Cauchy problems (parameterized by \( \xi \in \mathbb{R} \), of diffusive type thanks to the sector condition on \( \gamma^\pm \):
\[ \partial_x \psi^+(x, \xi) = -\gamma^+(\xi)\psi^+(x, \xi) + u(x) \quad \forall x \in [0, 1], \quad \psi^+(0, \xi) = 0 \] (2)

and
\[ \partial_x \psi^-(x, \xi) = \gamma^-\xi \psi^-(x, \xi) + u(x) \quad \forall x \in [0, 1], \quad \psi^-(1, \xi) = 0. \]

We shall say that a causal operator \( P^+ \) (resp. anti-causal operator \( P^- \)) admits a diffusive \( \gamma^+ \)-realization (resp. \( \gamma^- \)-realization) if there exists a so-called diffusive symbol \( \mu^+ (x, \xi) \) (resp. \( \mu^- (x, \xi) \)) so that \( P^+ u(x) = \langle \mu^+, \psi^+(u) \rangle \) (resp. \( P^- u(x) = \langle \mu^-, \psi^-(u) \rangle \)). Similarly, we say that an operator \( P \) admits a \( \gamma^\pm \)-diffusive realization if both its causal and anti-causal parts \( P^+ \) and \( P^- \) admit a diffusive realization associated respectively to \( \gamma^+ \) and \( \gamma^- \).

Let us state some sufficient conditions for the existence of the so-called canonical diffusive realization of an operator \( P \) for general paths \( \gamma^\pm \). They pertain to the Laplace transforms with respect to \( y \) of the causal and anti-causal parts of the impulse response: \( P^+(x, .) := \mathcal{L}(\tilde{p}(x, .)) \) and \( P^-(x, .) := \mathcal{L}(\tilde{p}(x, - .)) \).

**Theorem 1** For a given path \( \gamma^+ \) (resp. \( \gamma^- \)), a causal (resp. anti-causal) operator \( P^+ \) (resp. \( P^- \)) admits a diffusive symbol if the two following conditions are fulfilled:

(i) the Laplace symbol \( \lambda \mapsto P^+(x, \lambda) \) (resp. \( \lambda \mapsto P^-(x, \lambda) \)) is holomorphic in a domain \( D^+ \) (resp. \( D^- \)) that contains the closed set located at right of the arc \( -\gamma^+ \) (resp. at left of the arc \( -\gamma^- \));

(ii) \( \mathcal{P}^\pm(x, \lambda) \) vanish when \( |\lambda| \to \infty \) uniformly with respect to arg \( \lambda \).

Then the so-called canonical \( \gamma^\pm \)-symbols are given by
\[ \mu^+(x, \xi) = -\frac{\gamma^+(\xi)}{2i\pi} \mathcal{P}^+(x, -\gamma^+(\xi)) \] and \[ \mu^-(x, \xi) = \frac{\gamma^-\xi}{2i\pi} \mathcal{P}^-(x, -\gamma^-(\xi)) \] (3)

and have the same regularity as \( \gamma^\pm \).

### 3 Diffusive symbolic formulation of linear partial differential operational equations

In this section we state the equations satisfied by the symbols \( \mu^\pm \) equivalent to a boundary value problem posed on the kernel \( p \) in \( \Omega^+ \cup \Omega^- \), where the sets \( \Omega^\pm \) are associated to the causal \((y < x)\) and anti-causal \((y > x)\) parts of \( \Omega \). For this purpose we consider a partition of the boundary of \( \Omega^\pm \) in \( \Gamma^+_y = \{1\} \times \omega, \Gamma^-_y = \{0\} \times \omega, \Gamma_0 = \{(x, y) \in \Omega \text{ s.t. } x = y\}, \Gamma^+_x = \omega \times \{1\} \) and \( \Gamma^-_x = \omega \times \{0\} \).

We denote \( \nabla := \nabla (\partial_x, \partial_y) \) and consider a boundary value problem on the kernel \( p \):
\[ A(x, \nabla) p(x, y) = q(x, y) \text{ in } \Omega^+ \cup \Omega^- \] (4)

with an unspecified number of boundary conditions
\[ B(x, \nabla) p(x, y) = r(x, y) \text{ on } \partial \Omega^+ \cup \partial \Omega^-, \] (5)

where \( q \) is the kernel of a given operator \( Q \) with diffusive symbol \( \nu^\pm \). The restrictions of \( r \) to the boundaries \( \Gamma^+_y \) are assumed to be the kernels of a causal operator \( R^+ \) and an anti-causal
operator $R^-$ with diffusive symbols $\rho^+$ and $\rho^-$. With $\phi^\pm(x, y) := (x, \pm(x - y))$, the partial differential equation solved by $\tilde{\phi}^\pm := p \circ \phi^\pm$ is

$$\tilde{A}^\pm(x, \nabla) \tilde{\phi}^\pm = \tilde{q}^\pm$$

(6)

where $\tilde{A}^\pm(x, \nabla) = A(\phi^\pm(x, y), K^\pm \nabla)$ and $K^\pm := \begin{pmatrix} 1 & \pm1 \\ 0 & \mp1 \end{pmatrix}$. Recall that $\tilde{p}^\pm$ has analytic continuation on $\mathbb{R}^{+\times}$; we assume that this is also the case for $\tilde{A}$ (with respect to $y$). Furthermore, we can extend $\tilde{A}^\pm$ and $\tilde{p}^\pm$ to $\mathbb{R}^-$ by 0; so, the formulation of (6) in the sense of distributions takes the form:

$$\tilde{A}^\pm(x, \nabla) \tilde{p}^\pm + \sum_k \tilde{A}^\pm_k(x, \nabla) \tilde{p}^\pm \delta_0^{(k)} = \tilde{q}^\pm \text{ in } \mathcal{D}'(\mathbb{R})$$

(7)

where $\tilde{A}^\pm_k(x, \nabla)$ are suitable partial differential operators and $\delta_0^{(k)}$ is the $k$th derivative of the Dirac distribution at point 0. We are now in position to introduce two differential operators associated to $A$ and $\gamma^\pm$:

$$A^\pm(x, \partial_x, \lambda) = A(x, K^\pm \lambda (\partial_x, \lambda))$$

and

$$A_0^\pm(x, \nabla, \gamma^\pm(\xi)) = \frac{\gamma^\pm(\xi)}{2\pi} \sum_k (-\gamma^\pm(\xi))^k \tilde{A}_k^\pm(x, K^\pm \nabla);$$

(9)

Operators $B^\pm$ and $B_0^\pm$ can be derived from $B$ in the same way as $A^\pm$ and $A_0^\pm$ was from $A$.

**Theorem 2** Assuming that $P$, $Q$ and $R^\pm$ fulfil the assumptions of theorem 1, the kernel $p$ is solution of the boundary value problem (4-5) iff its canonical $\gamma^\pm$-symbols are solution of:

$$A^\pm(x, \partial_x, \gamma^\pm(\xi)) \mu^\pm(x, \xi) + A_0^\pm(x, \nabla, \gamma^\pm(\xi)) p(x, x) = \nu^\pm(x, \xi) \forall (x, \xi) \in \omega \times \mathbb{R}^+,$$

$$B^\pm(x, \partial_x, \gamma^\pm(\xi)) \mu^\pm(x, \xi) + B_0^\pm(x, \nabla, \gamma^\pm(\xi)) p(x, x) = \rho^\pm(x, \xi) \forall (x, \xi) \in \{0\} \times \mathbb{R}^+,$$

$$\left\langle B^\pm(x, \partial_x, \gamma^\pm(\xi)) \mu^\pm(x, \xi), e^{-\gamma^\pm(\xi)(x-y_0(x))} \right\rangle = r(x, y_0(x)) \text{ on } \Gamma_x \cup \Gamma_0$$

with $y_0(x) = x$, 0 or 1 on $\Gamma_0$, $\Gamma^+$ or $\Gamma^-$. For ending, we state some sufficient conditions on the operators $A$ and $B$ which insure that $P$ satisfies the assumption (i) of theorem 1.

The differential operators $A^\pm$ and $B^\pm$ can be expanded with respect to the derivatives: $A^\pm(x, \partial_x, \lambda) = \sum_m a^\pm_m(x, \lambda) \partial_x^m$ and $B^\pm(x, \partial_x, \lambda) = \sum_m b^\pm_m(x, \lambda) \partial_x^m$, which allows to define the union of zeros of the analytic functions $\lambda \mapsto a^\pm_m(x, \lambda)$ and $\lambda \mapsto b^\pm_m(x, \lambda)$ over all $x$ and $m$:

$$W^\pm_A := \bigcup_{x,m} [a^\pm_m(x, \cdot)]^{-1}(0) \text{ and } W^\pm_B := \bigcup_{x,m} [b^\pm_m(x, \cdot)]^{-1}(0).$$

**Theorem 3** If $W^\pm_A \cup W^\pm_B \subset \mathbb{C} - D^\pm$ and if $Q$ and $R^\pm$ fulfil the assumption (i) of theorem 1 then $P$ fulfils it also.
Final remarks:

1. The assumption (ii) of theorem 1 has to be checked case by case.

2. The general result has already been applied to two Lyapunov equations issued from the control theory of the heat equation with Dirichlet and Neuman boundary conditions.

3. The method has also been applied to the nonlinear Riccati equation related to a boundary control problem for the heat equation.

4. Numerical methods may be based or on the resolution of the boundary value problem satisfied by the kernel or directly on the differential equations of the diffusive symbols.

References


